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LETTER TO THE EDITOR

Scattering of charged particles off dyons

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Abstract. A recently developed group theoretical approach to scattering is applied to the modified Coulomb problem and thus by a purely algebraic manipulation the S matrix is derived.

Recent works by Alhassid *et al* (1983), Frank and Wolf (1984), Wu *et al* (1987) and the earlier paper of Barut and Kleinert (1967) have proved that, within the algebraic approach, the non-compact Lie groups can be as useful in describing scattering processes as the compact Lie groups have been in the treatment of bound-state problems. Such a possibility arises whenever the physical problem exhibits dynamical symmetry described by a Lie group G. In this case the scattering matrix S can be computed in a purely algebraic way by expanding the representations of G (describing the system in the presence of interactions) into representations of the group F (called the free or asymptotic group) describing the system in the absence of interactions (Iachello 1986).

Representations of G = (S-matrix) × representations of F. The process by means of which one goes from G to F is the contraction of the Lie group G. In the algebraic approach to scattering it is followed by an inverse process called an extension or Euclidean connection which amounts to the expansion of the generators of G into the enveloping algebra of F.

It is our purpose in this letter to apply this programme to the non-relativistic scattering of charged particles off a dyon (a particle which has both electric and magnetic charge) whose field is supplemented by a centrifugal potential.

More precisely, we consider the classical Hamiltonian system, (M, ω_μ, H_μ) where

$$\begin{aligned} &M = T^+R^3 = \{(x,p)\in R^3\times R^3;\ x\neq 0\} \\ &\omega_\mu = \mathrm{d}p\wedge \mathrm{d}x + \sigma_\mu \qquad \sigma_\mu = -\mu\left(\textstyle\sum \varepsilon_{ijk}x_i\,\mathrm{d}x_j\wedge x_k\right)r^{-3} \\ &H_\mu = \tfrac{1}{2}|P|^2 - \alpha/r + \mu^2/2r^2 \qquad r^2 = x_1^2+x_2^2+x_3^2 = |x|^2. \end{aligned} \tag{1}$$

McIntosh and Cisneros (1970), and later Iwai and Uwano (1986), have proved this system to possess higher (dynamic) symmetry than the obvious geometric symmetry. Namely, they found that under Poisson brackets

$$\begin{aligned} J_1 &= x_2p_3 - x_3p_2 + \mu x_1/r \qquad & Q_1 &= J_2p_3 - J_3p_2 + \alpha x_1/r \\ J_2 &= x_3p_1 - x_1p_3 + \mu x_2/r \qquad & Q_2 &= J_3p_1 - J_1p_3 + \alpha x_2/r \\ J_3 &= x_1p_2 - x_2p_1 + \mu x_3/r \qquad & Q_3 &= J_1p_2 - J_2p_1 + \alpha x_3/r \end{aligned} \tag{2}$$

are generators of symmetries of the Hamiltonian H_μ, i.e.

$$\{J_i, H_\mu\}_\mu = \{Q_i, H_\mu\}_\mu = 0 \qquad i = 1, 2, 3 \tag{3}$$

which span the Lie algebra

$$\{J_i, J_j\}_\mu = -\varepsilon_{ijk}J_k \qquad \{J_i, Q_j\}_\mu = -\varepsilon_{ijk}Q_k \qquad \{Q_i, Q_j\}_\mu = 2H_\mu\varepsilon_{ijk}J_k. \tag{4}$$

It turns out that it is isomorphic to one of the Lie algebras of the Lie groups SO(4), E(3) or SO(3, 1) and this depends on whether the fixed value E of H_μ is smaller than, equal to or greater than zero. A situation quite analogous to that arises in a many-body context (Solomon 1971) where the potential's change of sign switches from SU(2) to SU(1, 1). As far as we are concerned with scattering states $E = \frac{1}{2}k^2 > 0$ we can introduce $K_i = Q_i/\sqrt{2E}$ and write down the quantum analogues of (4) (via commutators) to arrive at the Lie algebra of the Lorentz group:

$$[\boldsymbol{J}_i, \boldsymbol{J}_j] = \mathrm{i}\varepsilon_{ijk}\boldsymbol{J}_k \qquad [\boldsymbol{J}_i, \boldsymbol{K}_j] = \mathrm{i}\varepsilon_{ijk}\boldsymbol{K}_k \qquad [\boldsymbol{K}_i, \boldsymbol{K}_j] = -\mathrm{i}\varepsilon_{ijk}\boldsymbol{J}_k.$$

Their irreducible representations (cf Naimark 1964) are labelled by the pairs (s, c) connected with the values of Casimir invariants

$$C_1 = \boldsymbol{J}^2 - \boldsymbol{K}^2 = s^2 + c^2 - 1$$

$$C_2 = \boldsymbol{J}\cdot\boldsymbol{K} = \mathrm{i}sc \qquad 2s \in Z,\ c \in C.$$

In our case $C_1 = \mu^2 - \alpha^2/k^2 - 1$ and $C_2 = \alpha\mu/k$ specify the equivalent representations $(\mu, -\mathrm{i}\alpha/k)$, $(-\mu, \mathrm{i}\alpha/k)$ of the so-called principal series of UIR of the Lorentz group. For defineteness we shall work further with the representation $(\mu, \mathrm{i}\rho)\rho = -\alpha/k$ supposing that $\mu > 0$. The action of generators of the Lorentz group on the basis $|1m\rangle$ of the Hilbert space of the representation $(\mu, \mathrm{i}\rho)$ is given in table 1.

Table 1.

$J_3\|l, m\rangle = m\|l, m\rangle$	$J_\pm = J_1 \pm \mathrm{i}J_2$
$J_+\|l, m\rangle = \alpha^l_{m+1}\|l, m+1\rangle$	$\alpha^l_m = [(l+m)(l-m+1)]^{1/2}$
$J_-\|L, m\rangle = \alpha_m\|l, m-1\rangle$	$l = \mu, \mu+1, \ldots$

$K_3|l, m\rangle = D_l(l^2 - m^2)^{1/2}|l-1, m\rangle - C_l m|l, m\rangle - D_{l+1}[(l+1)^2 - m^2]^{1/2}|l+1, m\rangle$

$K_+|l, m\rangle = D_l[(l-m)(l-m-1)]^{1/2}|l-1, m+1\rangle - C_l[(l-m)(l+m-1)]^{1/2}|l, m+1\rangle + D_{l+1}[(l+m+1)(l+m+2)]^{1/2}|l+1, m+1\rangle$

$K_-|l, m\rangle = -D_l[(l+m)(l+m-1)]^{1/2}|l-1, m-1\rangle - C_l[(l+m)(l-m+1)]^{1/2}|l, m-1\rangle - D_{l-1}[(l-m+1)(l-m+2)]^{1/2}|l+1, m-1\rangle$

$C_l = -\mu\rho/l(l+1)$

$D_l = \mathrm{i}[(l^2 - \mu^2)(l^2 + \rho^2)/(4l^2 - 1)]^{1/2}/l$

The general procedure for the contraction of a non-compact, connected semisimple Lie group G with a maximal compact subgroup M goes as follows. First, fix a Cartan decomposition of the Lie algebra of the Lie group G $\mathscr{J} = m \oplus k$, where m is the Lie algebra of M and $[m, k] \subset k$, $[k, k] \subset m$. Second, the contraction in the sense of Inönü and Winger is a limiting process: $m \to m$, $k \to p = \varepsilon k$, where $\varepsilon \to 0$. So in this manner one arrives at the new algebra $f = m \oplus p$ with $[m, p] \subset p$ and $[p, p] = 0$. The corresponding Lie group F is a semidirect product of its subgroups M and P, i.e. F = [P]M, P being an invariant subgroup in F. In our case the contraction of SO(3, 1) with respect to SO(3) obviously gives the group E(3). The generators of the latter act in the same

Hilbert space, the only difference being the exchange of the generators K_i and coefficients C_l, D_l with P_i, C'_l, D'_l:

$$C'_l = -\mu(\varepsilon\rho)/l(l+1) \qquad D'_l = \mathrm{i}[(l^2-\mu^2)(\varepsilon^2 l^2+\varepsilon^2\rho^2)/(4l^2-1)]^{1/2}/l. \tag{5}$$

In order to fall among the principal series of UIR of the Euclidian group the limit must be taken with care. Because if we let ε go to zero keeping μ and ρ fixed, then C'_l and D'_l tend to zero with ε and we get a representation which is reducible into the representations $l=\mu$, $l=\mu+1,\ldots$ of the rotation algebra. The other possibility, as pointed out by Voisin (1967), is to let ρ go to $\pm$ infinity (strong coupling $\alpha\to\mp\infty$) in such a way that

$$\lim_{\substack{\rho\to\pm\infty\\ \varepsilon\to 0}} \varepsilon\rho = \varkappa \tag{6}$$

with $\varkappa$ a fixed finite number. For physical reasons we choose $\varkappa=\pm k(\varepsilon=\mp k^2/\alpha)$ and then

$$C_l^{\mp} = \pm k\mu/l(l+1) \qquad D_l^{\mp} = -k[(l^2-\mu^2)/(4l^2-1)]^{1/2}/l \tag{7}$$

defines the $\pm k$ representations of the Euclidean group. In the contraction process the so(3, 1) Casimir invariants become the Casimir invariants of e(3)

$$C_1^{\mp} = \boldsymbol{P}^2 = k^2 \qquad C_2^{\mp} = -\boldsymbol{J}\cdot\boldsymbol{P} = \pm k\mu. \tag{8}$$

From the group theoretical point of view μ has an interpretation as 'spin' or 'helicity' within the Lorentz or Euclidean groups, respectively. Thus, the scattering may be seen as a helicity-flip process. A more precise discussion of this point can be found in Elliott and Dawber (1979).

The reverse concept of contraction-expansion (Rosen 1966), extension (Sankaranarayanan 1968) or Euclidean connection (Alhassid *et al* 1986) is a powerful technique which, applied in a systematic fashion, can be useful in much more general scattering problems. Here we shall need only the asymptotic expansion of K_i written in terms of the generators of E(3) as

$$\begin{aligned} K_\pm &= \mp\lambda P_\pm - \mathrm{i}[P_\pm \mp (P_\pm J_3 - P_3 J_\pm)]/k \\ K_3 &= \pm\lambda P_3 + \mathrm{i}[-P_3 + (P_+J_- - P_-J_+)/2]/k. \end{aligned} \tag{9}$$

The sign before λ in K_3 depends on whether we work in the $+k$ or $-k$ representation of E(3). More details can be found in Frank *et al* (1986).

On the other hand, the value of the variable parameter λ is fixed by the requirement that the operators from (9) define (asymptotically) the representation $(\mu, \mathrm{i}\rho)$ of the Lorentz group. This is achieved when λ is chosen to be ρ/k.

Acting on both sides of

$$|\rho,\mu,l,m\rangle = A_l(k)|-k,\mu,l,m\rangle + B_l(k)|k,\mu,l,m\rangle \tag{10}$$

with K_3 we get the recursion relations for the reflection amplitude $R_l(k)=B_l(k)/A_l(k)$ in the form

$$R_{l+1}(k) = -\frac{l+1+\mathrm{i}\alpha/k}{l+1-\mathrm{i}\alpha/k}R_l(k). \tag{11}$$

From these, for the scattering matrix $S_l(k) = \exp[\mathrm{i}(l+1)\pi]R_l(k)$ one obtains

$$S_l(k) = \frac{\Gamma(l+1+\mathrm{i}\alpha/k)}{\Gamma(l+1-\mathrm{i}\alpha/k)} \qquad l \geqslant \mu. \tag{12}$$

The cross section arising from (11) is given by

$$\mathrm{d}\sigma/\mathrm{d}\Omega = |f(k, \theta)|^2 \tag{13}$$

where

$$f(k, \theta) = (2\mathrm{i}k)^{-1} \sum_{l \geqslant \mu} (2l+1)(S_l(k) - 1)\, \mathrm{d}^l_{-\mu\mu}(\theta). \tag{14}$$

Inserting the last expression into (13) finally produces the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2 + \mu^2 k^2}{k^4 \sin^4 \frac{1}{2}\theta}$$

which has identical angular dependence as the classical result of Rutherford.

Working on an apparently different problem—the scattering of Bogomolny-Prasad-Sommerfeld monopoles—Gibbons and Manton (1986) and Feher and Horvathy (1987) arrived nevertheless at very similar results.

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